ISP220, fall 2021: In-Class Project #5; 15 pts, Plus 10 points bonus!

Quarks, Spacetime, and the Big Bang

Tuesday, September 21, 2021

Name: KEY Student #

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1 Kepler's 3rd law & the International Space Station: 10 pts

Kepler's Third law said originally, that the square of the period of a planet is proportional to the cube of the planet's distance from the sun. For this original usage, let's call the proportionality constant, k_s and this becomes the equation:

$$T^2 = k_s R^3 \tag{1}$$

The same relationship holds for objects orbiting the earth, (or any gravitationally bound system) except now the constant has a different value (stay tuned for Newton). Let's call that relationship:

$$T^2_{ISS} = k_e R^3_{ISS} \tag{2}$$

What we're going to do is figure out how long it takes for the International Space Station (ISS) to orbit the earth.

We'll need Equation 2, but we have to find k_e...or do we. Inquiring minds want to know. For that...our favorite earth-orbiting object comes in handy: the moon. The value of k_e for the moon would be the same value for anything orbiting earth. So let's figure out how to make use of it. Here are relevant data for the moon:

1. Radius of the moon's orbit from the center of the earth, $R_m = 385 \times 10^3$ km.
2. The time that it takes for the moon to orbit the earth, its period is $T = 27.3$ days.

Let's suggestively give names to the variables in Equation 2.

$$T^2_m = k_e R^3_m \tag{3}$$

Figure 1: The figure below is a picture of the Soyuz MS-01 spacecraft docked to the ISS in July of 2016. The crew consisted of Anatoli Ivanishin (Russia), Takuya Onishi (Japan), and Kathleen Rubins (USA). Scott Kelly (now he's running for the senate in Arizona) had returned from his year on the ISS the previous March. There have been roughly 60 Soyuz missions to ISS. A Soyuz capsule is docked all the time as an escape vehicle.

Now let's consider the ISS. Here are data for it:

1. Altitude of the ISS is $h_{ISS} = 400$ km.
2. Radius of the earth is $R_e = 6380$ km.
3. The period of the ISS? That's what we're going to figure out.

What is the radius of the ISS' orbit from the center of the earth?

(your work:)

$R_{ISS} = R_E + h = 6380 + 400 = 6780$

Answer 3 points: $R_{ISS} =$ 6780 km

The trick is that we don't actually need to calculate k_e. Since it's the same in both Equation 2 and 3 we can equate them, leaving an equation consisting of $R_m^3, R_{ISS}^3, T_m^2,$ and T_{ISS}^2.

What is that relation? Inside the box write an equation with three terms:

$k_e =$ a ratio for the moon $=$ a ratio for the ISS

$T^2 = k R^3 \Rightarrow k = T^2 / R^3$

Answer 3 points: $k_e = \dfrac{T_m^2}{R_m^3} = \dfrac{T_{ISS}^2}{R_{ISS}^2}$

From the last two terms, you can solve for T_{ISS}. Do that and plug in the numbers and calculate T_{ISS} in the Big Box below. (Hint: it's less than a tenth of a day.)

Remember that you can use Mr Google as a calculator for arithmetic and unit conversions, but also for evaluating equations. How to use Google as a calculator

(all of your work...write symbols before inserting numbers:)

$$T_{ISS}^2 = \frac{T_m^2}{R_m^3} R_{ISS}^3 = T_m^2 \left(\frac{R_{ISS}}{R_m}\right)^3$$

$$= (27.3)^2 \left(\frac{6780\text{ km}}{385\times10^3\text{ km}}\right)^3$$

$$T_{ISS} = \sqrt{(27.3)^2\,(1.8\times10^{-2})^3}$$

$$= 6.4\times10^{-2}\text{ d}$$

Answer 6 points: $T_{ISS} =$ 6.4×10^{-2} days 0.064

How many hours is that? (Should be less than a couple.)

$\times 24$

Answer 3 points: $T_{ISS} =$ 1.5 h

2 Searching for Exoplanets: Bonus 15 points

Let's go back to Kepler's original use: the solar system with planets orbiting the sun.

$$T^2 = k_s R^3 \tag{4}$$

This function is plotted on Figure 2 for you. If you don't remember how to read log plots,

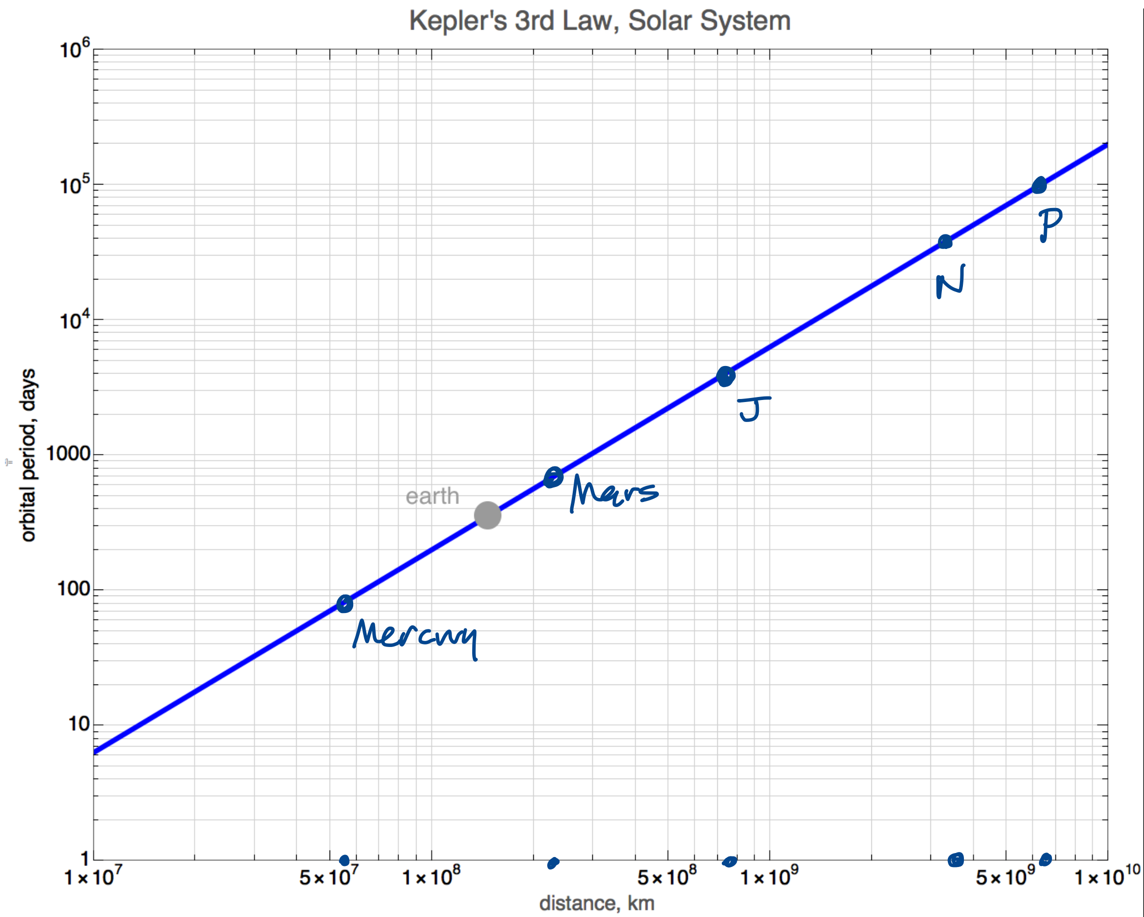


Figure 2: Kepler's 3rd Law represented by the Period (T) in days plotted against the distance a planet is from the sun (R) in km.

you can look at Section 4.8.1, "Log-Log and Semi-Log Plots" in Lesson 4. Just remember: there's no zero, rather the major lines are 1, 2, 3...10 times whatever power of 10 is relevant for that interval. The following will be approximate answers given the resolution of the plots. Here are some numbers for the planets:

1. From the data in Figure 3, put a big dot on the graph in Figure 2 at the period for

Celestial Object	Mean Distance from Sun (million km)
SUN	—
MERCURY	57.9
VENUS	108.2
EARTH	149.6
MARS	227.9
JUPITER	778.4
SATURN	1,426.7
URANUS	2,871.0
NEPTUNE	4,498.3
EARTH'S MOON	149.6 (0.386 from Earth)

Figure 3: Distance from the sun for the planets in millions of kilometers. Notice that the distance from the sun to earth is 149×10^6 km

mercury, mars, jupiter, and neptune and label each one. I've done earth for you. (4 points)

2. Poor pluto is no longer a planet, but it's still a part of the family. It's mean distance from the sun is about 39 AU, where 1 AU ("Astronomical Unit") is the mean distance of the earth to the sun. So it's about $39 \times 149 \times 10^6$ km from the sun. Put its position on the graph and notice the period in days for pluto to orbit the sun. (1 point)

3. Pluto was discovered by Clyde W. Tombaugh at the Flagstaff Observatory in 1929. Roughly what fraction of Pluto's orbit has occurred in the 91 years (=33,220 days) since it was discovered?

$R_p = 5.8 \times 10^9$ km $\quad f = \frac{T_P}{33{,}220} = \frac{9. \times 10^4}{3.3 \times 10^4} \simeq 3$

Answer (1 point): Fraction of pluto's orbit in 91 years = 1/3

Newton's Explanation of Kepler's Third Law
Newton showed that Kepler's 3rd Law follows directly from his Universal law of Gravitation as you'll see in Lesson 10. A general version of Kepler's 3rd Law replaced k_s with constants involving the mass of the central body, whether the sun, or the earth, or some other star. That's why there's a different k_e from k_s. So for earth, as an example, Newton would say:

$$T_e^2 = \frac{4\pi^2}{GM_s} R_e^3 \tag{5}$$

So the quantity $\frac{4\pi^2}{GM_s}$ is the constant k_s. G is the Gravitational Constant...see Lesson 10.

So there are many little Copernican/Keplerian systems in our neighborhood. The moon orbiting the earth is one as you know from the first question. The moons of Jupiter clued Galileo in on the fact that they appeared to be little planetary systems obeying Kepler's 3rd law. But that's not all:

Exoplanets

A more modern situation is the search for Exoplanets...planets in systems under the influence of stars other than our sun. That's the job of the Kepler Telescope (you now know why it was named after our Johannes Kepler) and it uses the following technique:

The Kepler telescope looks at the light from stars and when the light dims and then comes back...and then dims again...and comes back—and it does so repeatedly and with the same time between dips, it's assumed that a planet has passed in front of the star, and reduced the light that makes its way to the telescope.

Here's an example

So far, Kepler has discovered 2600 or so planets with another 3000 being still evaluated. This mission is a sterling success for NASA and continued years beyond its originally scheduled end-point because, like its namesake, the satellite just kept going and going and going... It was launched in 2009 and decommissioned in 2018. It was expected to last for one year. Pretty good.

Kepler discoveries are named this way: Kepler-Nx would be from the Nth star that Kepler found having the xth planet, where x starts with the letter b. So Kepler-11g, is the 6th planet found around Kepler's 11th star (`https://exoplanetarchive.ipac.caltech.edu/docs/counts_detail.html`) .

The Mass of Kepler-7b's Star

Let's take one of the early Kepler discoveries and calculate the mass of the star around which "Kepler-7b" was found in 2010. We now know that this planet is bigger than Jupiter and that its star is near the end of its life. But how big is that star?

We can use Equation 5 modified for our particular exoplanet:

$$T_{7b}^2 = \frac{4\pi^2}{GM_7} R_{7b}^3 \tag{6}$$

$$T_{7b}^2 = k_{7b} R_{7b}^3 \tag{7}$$

where now we're talking about an arbitrary new sun, the sun with mass of M_7 and its planet at a distance R_{7b} and period T_{7b} from it.

We can take what we know, namely Equation 6 and combine it with the equation specifically suited to our sun, where I've chosen the earth's distance and period:

$$T_e^2 = \frac{4\pi^2}{GM_S} R_e^3 \tag{8}$$

Notice that there are factors in common (2, π, G) between Eqs. 6 and 8 and so we can manipulate them to find a relationship between the mass M of any gravitational system (here, M_7) as compared with M_S from our system. That relationship turns out to be:

$$\frac{M_7}{M_S} = k_s \frac{R_{7b}^3}{T_{7b}^2} \tag{9}$$

I've calculated and used $k_s = 2.3 \times 10^{-30} \mathrm{d}^2/\mathrm{km}^3$ and the radius of the candidate planet Kepler-7b's orbit of $R_{7b} = 9,350,000$ km and so the graph in Figure 4 is explicitly M/M_s versus T_{7b} days:

$$\frac{M_7}{M_s} = \frac{k_s}{R_{7b}} T_{7b} = \frac{2.3 \times 10^{-30}}{9,350,000} T_{7b} = 2.46 \times 10^{-37} T_{7b}. \tag{10}$$

So a measurement of the period, T_{7b} of our Exoplanet and R_{7b} the radius of the orbit will give us the ratio of the mass of its sun to our sun. Impressive, right? The light curve is shown in Figure 5.

1. Find the period in days of Kepler-7b from Figure 5 and put a dot on the horizontal axis at that value (3 points).
2. Then in Figure 4 draw the vertical line up from the horizontal axis to the curve and then over to the vertical axis. (2 points)
3. What is the ratio of Kepler-7b's favorite star to the mass of our sun?

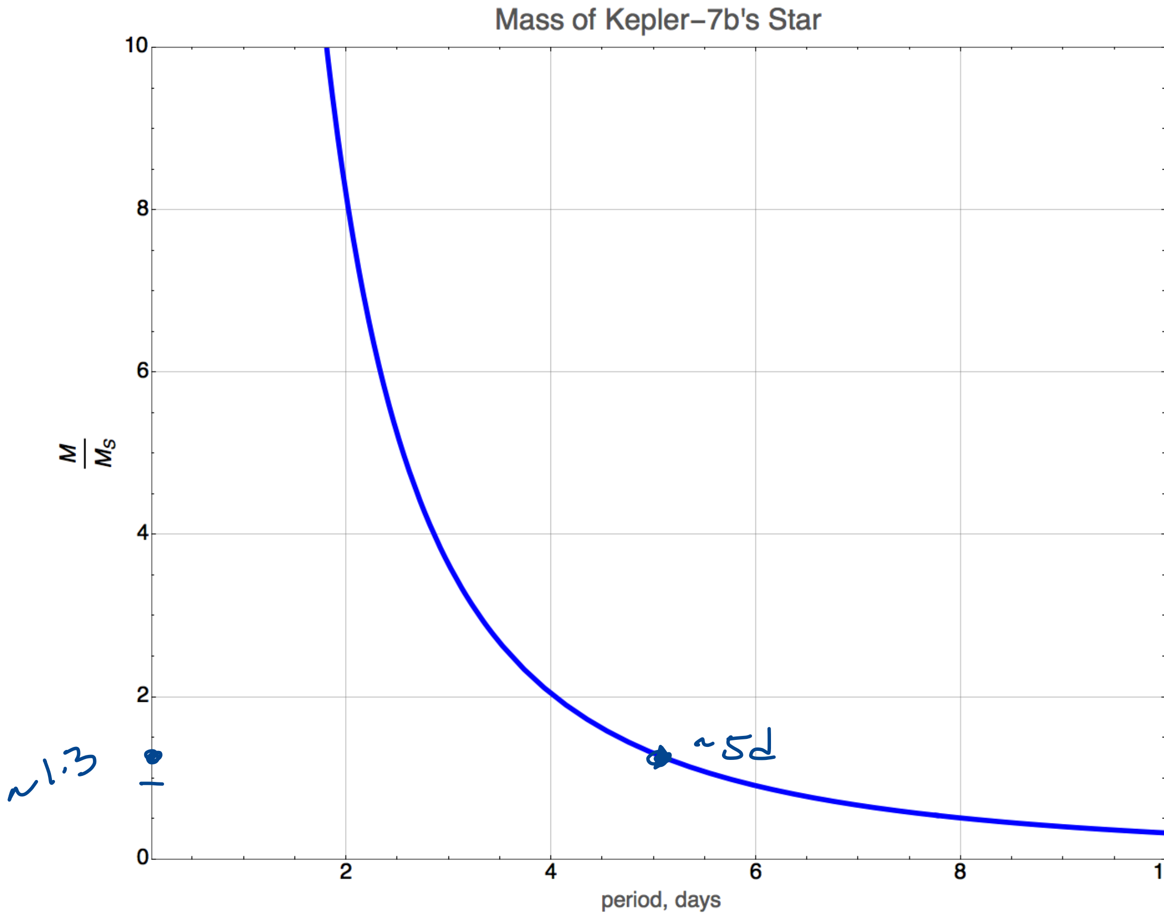
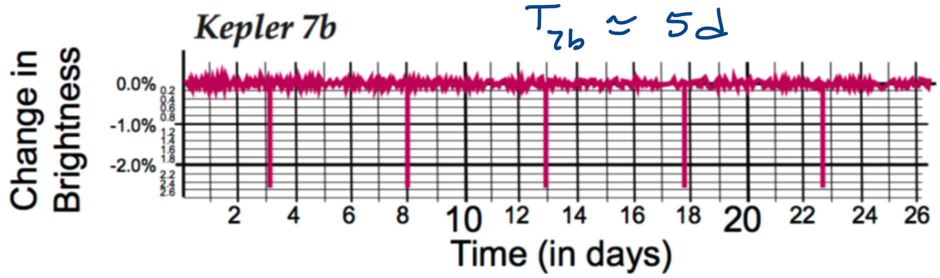


Figure 4: The ratio of the Kepler 7b central star versus the period of Kepler 7b.

Answer (4 points): $\dfrac{M(\text{ Kepler-7b's star})}{M_S} = $ 1.3



$T_{7b} \simeq 5d$

Figure 5: The light curve for Kepler-7b. Notice the light repeatedly —and briefly —dims as the planet passes in front.