9. Quantum Statistics, 2

lecture 32, November 13, 2017

housekeeping

Honors project

- Instructions for 3 people are up
  - *that was easy*

This week, remember:

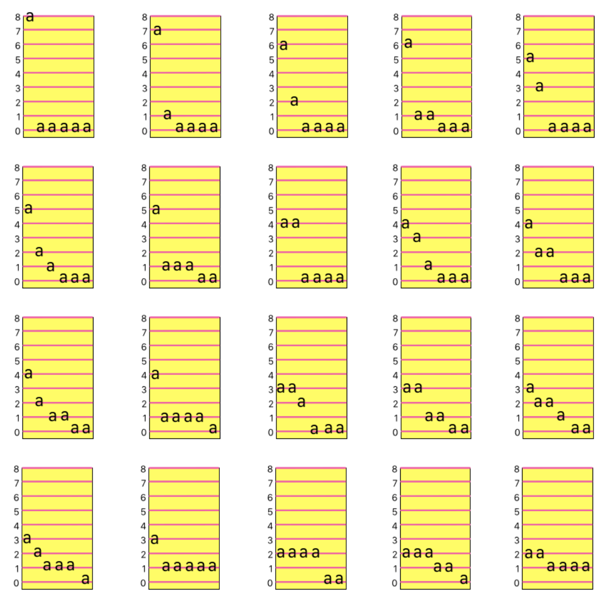
- homework workshop will be Wednesday
- homework will be due Friday
- lectures will happen M,T,F

today

statistical physics - cla~~ssically~~ speaking

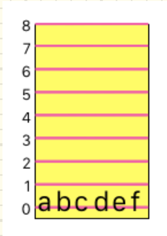
quantum mechanically

Last Friday: game of 6 molecules, 9 energy states: goal to count # ways to make $E_{TOTAL} = 8$

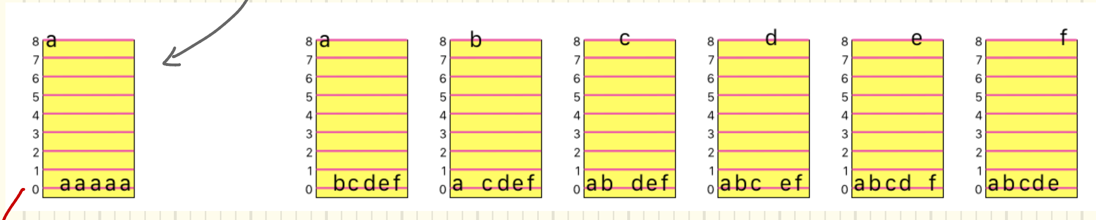


← 20

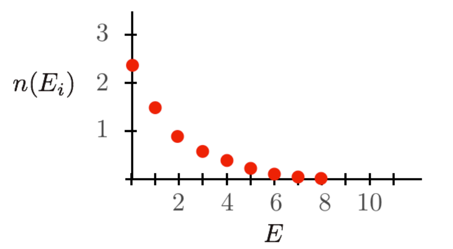
But now let the 6 molecules be distinguishable:



Distinguishability took to



	energy slot value									
Mac(i)	0	1	2	3	4	5	6	7	8	# micro
1	5	0	0	0	0	0	0	0	1	6
2	4	1	0	0	0	0	0	1	0	30
3	4	0	1	0	0	0	1	0	0	30
4	4	0	0	1	0	1	0	0	0	30
5	4	0	0	0	2	0	0	0	0	15
6	3	2	0	0	0	0	1	0	0	60
7	3	0	2	0	1	0	0	0	0	60
8	3	0	1	2	0	0	0	0	0	60
9	3	1	1	0	0	1	0	0	0	120
10	3	1	0	1	1	0	0	0	0	120
11	2	0	4	0	0	0	0	0	0	15
12	2	2	0	2	0	0	0	0	0	90
13	2	1	2	1	0	0	0	0	0	180
14	2	2	1	0	1	0	0	0	0	60
15	2	3	0	0	0	1	0	0	0	30
16	1	4	0	0	1	0	0	0	0	120
17	1	3	1	1	0	0	0	0	0	60
18	1	2	3	0	0	0	0	0	0	60
19	0	4	2	0	0	0	0	0	0	15
20	0	5	0	1	0	0	0	0	0	6
$n(E)$	2.31	1.54	0.98	0.59	0.33	0.16	0.07	0.02	0.005	1287



Then, using distinguishability as key ... from an ideal gas:

Maxwell velocity distribution $\longrightarrow$ Maxwell speed distribution

$$f(v_1) = \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m v_1^2}{2kT}}$$

$$n(v)\,dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT}\,dv$$

Then.. changed to energy:

$$n(E)\,dE = n\frac{2}{\sqrt{\pi}}\frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT}\,dE$$ ideal gas

$$n(E)\,dE = A\,e^{-E/kT}\,dE$$ general

$$F_{MB} = A\,e^{-E/kT}$$

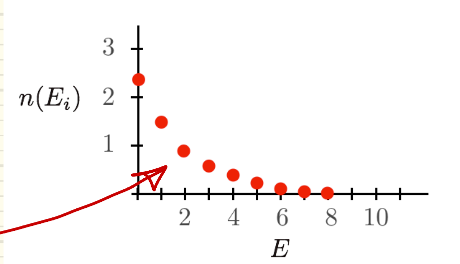
Maxwell-Boltzmann Distribution Function

$$n(E) = g(E)F_{MB}(E)$$ Density of States

number of ways a system can achieve an energy E or, discretely...

$$n(E_i) = g_i(E_i)\,F_{MB}(E)$$

$$F_{MB} = A\,e^{-E/kT}$$



The MB distribution is relevant for "classical" situations

$\downarrow$

indistinguishability not relevant

can still be quantum mechanical
in the sense of quantized
energy states...

In a stellar atmosphere of $T = 3000$ K .. what is the relative population of the ground state and first excited state for atomic hydrogen?

$$E_n = -\frac{13.6}{n^2}\ \text{eV} \qquad g_n = n^2 \qquad \begin{aligned} E_1 &= -13.6\ \text{eV} \\ E_2 &= -3.4\ \text{eV} \end{aligned}$$

$$n_1 = g_1 A e^{-E_1/kT}$$

$$n_2 = g_2 A e^{-E_2/kT}$$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\frac{e^{-E_2/kT}}{e^{-E_1/kT}} = \frac{4}{1} e^{-(13.6-3.4)/kT} = 4e^{-10.2/(8.617\times10^{-5})(3000)}$$

$$\frac{n_2}{n_1} = 4e^{-39} \simeq 3\times10^{-17}$$ $\Rightarrow$ overwhelmingly in ground state

When classical & when quantum mechanical?

MB applies

something else applies

Take our gas again

$$\tfrac{3}{2} kT = \tfrac{1}{2} \frac{p^2}{2m} \xrightarrow{QM} p = \frac{h}{\lambda}$$

if $\lambda <<$ average distance between molecules $\Rightarrow$ classical should be good.

$$p = \sqrt{3mkT}$$

so $$\lambda = \frac{h}{\sqrt{3mkT}} << \sqrt[3]{\frac{V}{N}} \quad \text{... ish}$$

So when: $$\left(\frac{N}{V}\right) \frac{h^3}{(3mkT)^{3/2}} << 1$$

we would expect MB to be okay.

high T and/or small m

Can MB statistics be used for hydrogen gas @ STP?

$$\text{STP} \Rightarrow \quad N_{mol} = N_A = 6.02 \times 10^{23} \text{ molecules}$$

$$\# \text{ in } V = 22.4\, l \quad = 22.4 \times 10^{-3}\, m^3 \quad \& \ T = 300\, K$$

$$m_{H_2} = 3.34 \times 10^{-27}\ kg.$$

$$\left(\frac{N}{V}\right) \frac{h^3}{(3mkT)^{3/2}} = \frac{(6.02 \times 10^{23})}{(22.4 \times 10^{-3})} \frac{(6.626 \times 10^{-34})^3}{(3)(3.34 \times 10^{-27})(1.4 \times 10^{-23})(300)}$$

$$= 3.37 \times 10^{-5} \quad \ll 1$$

So MB statistics can be used.

Go back to the Energy-box game. $\longrightarrow$ consider quantum mechanics.

distinguishability $\longrightarrow$ indistinguishability

For that $E=8$ state which had 6 distinguishable microstates... now we have just 1:

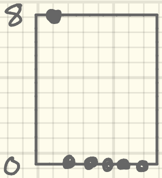
Mac(1)

is indistinguishable from the other 5

$\Rightarrow$ 20 distinguishable states, all equally likely

$\longrightarrow$ no restriction on number in any state $\Rightarrow$ BOSONS

Mac(1)



The average number of particles with $E=0$ for Mac(1) = 5

likelihood of Mac(1)

$$\langle n_0(1)\rangle = n_0(1)\,P_1$$

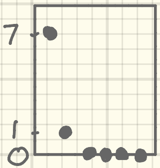
average # with $E=0$ for Mac(1)

$E=0$

in Mac(1)

$$\langle n_0(1)\rangle = 5\left(\frac{1}{20}\right) = 0.25$$

Mac(2)



$$\langle n_0(2)\rangle = 4\left(\frac{1}{20}\right) = 0.20$$

$$\langle n_0\rangle = \sum_{i=1}^{20}\langle n_0(i)\rangle = \sum_{i=1}^{20} n_0(i)\,P_i = 2.45$$

"Bose-Einstein"

E	n_{MB}	n_{BE}
0	2.31	2.45
1	1.54	1.55
2	0.98	0.90
3	0.59	0.45
4	0.33	0.30
5	0.16	0.15
6	0.07	0.10
7	0.02	0.05
8	0.005	0.05

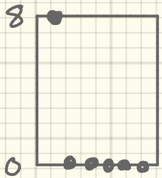
More quantum mechanics:

indistinguishable — Bosons

+

Pauli exclusion principle — Fermions : no state can have more than 2 $s=\frac{1}{2}$ particles

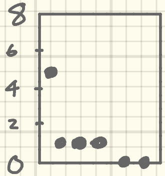
Mac(1)



is illegal

more than 1

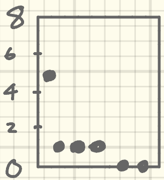
Mac(7)



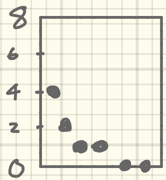
ditto

BUT!

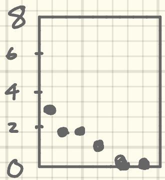
Mac(7)



Mac(14)



Mac(15)



A-OK...

the only ones

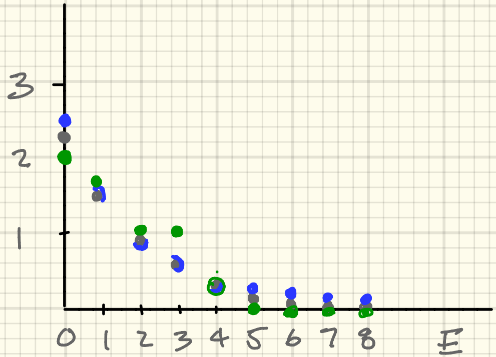
So $\langle n_0 \rangle_{FD} = \sum_{i=1}^{3} \langle n_0(i) \rangle = \sum_{i=1}^{3} n_0(i) P_i$

$$= 2\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right) = 2.0$$

Fermi-Dirac

Complete the table!

E	n_{MB}	n_{BE}	n_{FD}
0	2.31	2.45	2
1	1.54	1.55	1.67
2	0.98	0.90	1
3	0.59	0.45	1
4	0.33	0.30	0.33
5	0.16	0.15	0
6	0.07	0.10	0
7	0.02	0.05	0
8	0.005	0.05	0



Differences will be noticeable with more particles

Remember ψ_A and ψ_S

appropriate for FD quanta

appropriate for BE quanta

2 identical Bosons:

$$\psi_S = \sqrt{1/2}\left[\psi_\alpha(1)\,\psi_\beta(2) + \psi_\beta(1)\,\psi_\alpha(2)\right]$$

normalization b/c ψ_α & ψ_β are normalized.

α & β: all identifiers for "a state"

Put both in same state $\Rightarrow$ $\alpha = \beta$

$$\psi_s = \sqrt{\tfrac{1}{2}} \left[\psi_\beta(1)\psi_\beta(2) + \psi_\beta(2)\psi_\beta(1) \right]$$

$$\psi_s = \sqrt{2}\, \psi_\beta(1)\psi_\beta(2)$$

so $$\psi_s^* \psi_s = 2\, \psi_\beta^*(1)\psi_\beta^*(2)\psi_\beta(1)\psi_\beta(2)$$

Suppose indistinguishability <u>not</u> <u>required</u>

Then

$$\psi_D = \psi_\alpha(1)\psi_\beta(2)$$

same state: $$\psi_D^* \psi_D = \psi_\beta^*(1)\psi_\beta^*(2)\psi_\beta(1)\psi_\beta(2)$$

Notice:

$$P(\psi_s) = 2\, P(\psi_D)$$

How about 3 such particles?

$$\psi_s = \sqrt{1/6}\,\big[\,\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) + \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3)$$

$$+ \psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3) + \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3)$$

$$+ \psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) + \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\big]$$

6 terms -- OBTW $= 3!$

same state $\Rightarrow$ $\alpha = \beta = \gamma$

$$\psi_s = \sqrt{1/6}\cdot 6\;\psi_\beta(1)\psi_\beta(2)\psi_\beta(3)$$

$$\psi_s^*\psi_s = \underbrace{\frac{1}{6}\cdot 6^2}_{\frac{1}{3!}\cdot(3!)^2}\;\underbrace{\psi_\beta^*(1)\psi_\beta^*(2)\psi_\beta^*(3)\psi_\beta(1)\psi_\beta(2)\psi_\beta(3)}_{\psi_D^*\psi_D}$$

$$\psi_s^*\psi_s = 3!\,\psi_D^*\psi_D$$

an enhancement effect for indistinguishability of Bosons: $n!$

Think of it this way:

Imagine an empty state ... add a boson $\Rightarrow$ $P_1 =$ probability that it will land in state 1

Add more ... each would have same probability

for distinguishable quanta

$$P_n^D = \underbrace{P_1^D \cdot P_1^D \cdot P_1^D \cdots P_1^D}_{n \text{ times}}$$

$$= (P_1)^D$$

But:

For distinguishable - Boson - quanta

$$P_n^B = n!\, P_n^D = n!\,(P_1)^n$$

stay with me ...

The probability that there would be n+1

$$P^B_{n+1} = (n+1)!\, P_{n+1}$$

note $(n+1)! = (n+1)\,n!$ $\qquad (n+1)\underbrace{(n+1-1)(n+1-2)\cdots}_{n!}$

$$P^B_{n+1} = (n+1)\,n!\, P_{n+1}$$

and $P_{n+1} = (P_1)^{n+1} = (P_1)^n P_1 = P^D_n P_1$

so $P^B_{n+1} = (n+1)\underbrace{n!\; P^D_n}_{\downarrow\; P^B_n} P_1$

$$P^B_{n+1} = (n+1)\, P^B_n P_1 = (n+1)\, P_1 P^B_n$$

prob that there are n in state

prob of adding 1 D'ably.

$\Rightarrow$ if there are already n bosons in state probability that 1 more can add is $(n+1) \times P^D_1$

EQUILIBRIUM

Imagine a collection of distinguishable particles in thermal equilibrium

They each have energy... can't influence the others except in elastic collisions

→ consider 2 different energy states ϵ_1 and ϵ_2

Average number with ϵ_1: n_1

Average number with ϵ_2: n_2

$1 \xrightarrow{\text{can}} 2$ ... average rate R_{12} per particle

$2 \longrightarrow 1$ ... average rate R_{21} per particle

Total rate at which particles go $1 \to 2$: $n_1 R_{12}$

" $2 \to 1$: $n_2 R_{21}$

If that's all that can happen:

$$n_1 R_{12} = n_2 R_{21}$$

So then $$\frac{n_1}{n_2} = \frac{R_{2-1}}{R_{1-2}}$$

average # n_1 of particles to be in ϵ_1 is gotten from F_{MB}

$$n_1 = n(\epsilon_1) = A e^{-\epsilon_1/kT}$$

ditto $$n_2 = n(\epsilon_2) = A e^{-\epsilon_2/kT}$$

So

$$\frac{n_1}{n_2} = \frac{e^{-\epsilon_1/kT}}{e^{-\epsilon_2/kT}} = \frac{R_{21}}{R_{12}}$$

Remember the Boson enhancement.

Equilibrium:

$$n_1 R^B_{1\to 2} = n_2 R^B_{2\to 1}$$

relate to distinguishable form, D,

$$R^B_{1\to 2} = (1+n_2) R^D_{1\to 2}$$

$$R^B_{2\to 1} = (1+n_1) R^D_{2\to 1}$$

so!

$$n_1(1+n_2) R^D_{1\to 2} = n_2(1+n_1) R^D_{2\to 1}$$

$$\frac{n_1(1+n_2)}{n_2(1+n_1)} = \frac{R^D_{2\to 1}}{R^D_{1\to 2}} = \frac{e^{-\epsilon_1/kT}}{e^{-\epsilon_2/kT}}$$

$$\frac{n_1(1+n_2)}{n_2(1+n_1)} = \frac{R^D_{2\to 1}}{R^D_{1\to 2}} = \frac{e^{-\epsilon_1/kT}}{e^{-\epsilon_2/kT}}$$

$$\frac{n_1}{1+n_1} e^{+\epsilon_1/kT} = \frac{n_2}{1+n_2} e^{+\epsilon_2/kT} \equiv h(T)$$

let $h(T) = e^{-\alpha(T)}$

$$\frac{n_1}{1+n_1} e^{+\epsilon_1/kT} = h$$

$$\frac{n_1}{1+n_1} = h \cdot e^{-\epsilon_1/kT} = e^{-\alpha} e^{-\epsilon_1/kT}$$

$$n_1 = n_1 e^{-(\alpha+\epsilon_1/kT)} + e^{-(\alpha+\epsilon_1/kT)}$$

$$n_1\left[1 - e^{-(\alpha+\epsilon_1/kT)}\right] = e^{-(\alpha+\epsilon_1/kT)}$$

$$n_1 = \frac{e^{-(\alpha+\epsilon_1/kT)}}{1 - e^{-(\alpha+\epsilon_1/kT)}} = \frac{1}{e^{\alpha} e^{\epsilon_1/kT} - 1}$$ ditto n_2

$$n_B(E) = \frac{1}{e^{\alpha} e^{E/kT} - 1}$$

The Bose Einstien Distribution

book calls $e^{\alpha} = B_{BE}$

Fermions ...

same argumentation $\longrightarrow$ an inhibition factor

$$R^F_{1\to 2} = (1-n_2)\, R^D_{1\to 2}$$

$$R^F_{2\to 1} = (1-n_1)\, R^D_{2\to 1}$$

$\vdots$ same

$$n_F(E) = \frac{1}{e^{\alpha} e^{E/kT} + 1}$$

book calls $e^{\alpha} = B_{FD}$

see problem 9-21

Three Probability Distributions

$$F_{MB} = \frac{1}{A\, e^{E/kT}}$$

$E/kT >> 1$

tiny

$$F_{BE} = \frac{1}{B e^{E/kT} - 1} \longrightarrow F_{MB}$$

$$F_{FD} = \frac{1}{C e^{E/kT} + 1} \longrightarrow F_{MB}$$