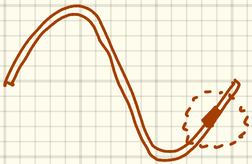
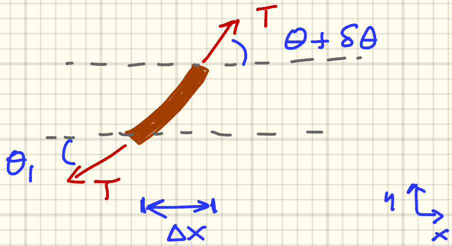
WAVES

Review (?) of mechanical waves

STRING THEORY ?



a little chunk



Short section: T constant at both ends, but angles slightly different

Vertically: $(F_{net})_y = T\sin(\theta + \delta\theta) - T\sin\theta$

for small θ $\quad \sin\theta \sim \tan\theta = \dfrac{\partial y}{\partial x} \rightarrow$ slope at point x

y goes up and down in time $\Rightarrow$ transverse velocity $\frac{\partial y}{\partial t}$

$$(F_{net})_y \cong T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

which makes you think "derivative" ...

$$\frac{(F_{net})_y}{\Delta x} = T\left\{\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x}\right\}$$

take limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2}$$

The y position is $y(x,t)$ ...

$$(F_{net})_y = T \frac{\partial^2 y}{\partial x^2} \Delta x$$

$$(F_{net})_y = T \frac{\partial^2 y}{\partial x^2} \Delta x$$

the mass of that chunk is $m = \mu \Delta x$

mass/length

From Newton's 2nd:

$$(F_{net})_y = \sum F_y = m a_y$$

$$T \frac{\partial^2 y}{\partial x^2} \Delta x = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

so

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \left(\frac{\mu}{T}\right) \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\left[\frac{\mu}{T_{en}}\right] = \frac{\left[M/L\right]}{\left[\frac{ML}{T_{ime}^2}\right]} = \frac{T_{ime}^2}{L^2} = \left[\frac{1}{v}\right]^2$$

so that term is the inverse of the $\updownarrow$ velocity

... for an actual, material wave, its $\left(\frac{\text{mass/length}}{\text{tension}}\right)$

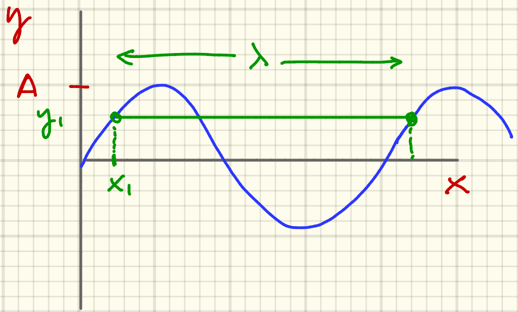
Ta Da:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ is THE WAVE EQUATION

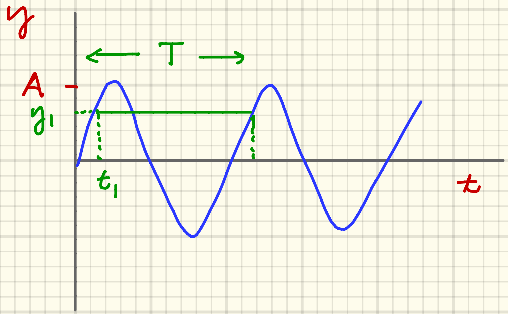
Real solutions: $y(x,t) = A \sin \frac{2\pi}{\lambda}(x \pm vt + \delta)$

phase depends on where $t=0$, set $\delta = 0$.

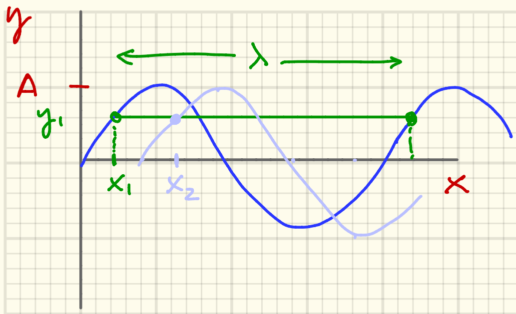
traveling waves $y(x,t) = A \sin \frac{2\pi}{\lambda}(x \pm vt)$



@ $t = t_1$



@ $x = x_1$



@ $t = t_1$

at some later time, the height position has moved .. choose − sign

$$y_1(x_1 t_1) = A \sin \frac{2\pi}{\lambda}(x_1 - v t_1)$$

to

$$y_2(x_2 t_2) = A \sin \frac{2\pi}{\lambda}(x_2 - v t_2)$$

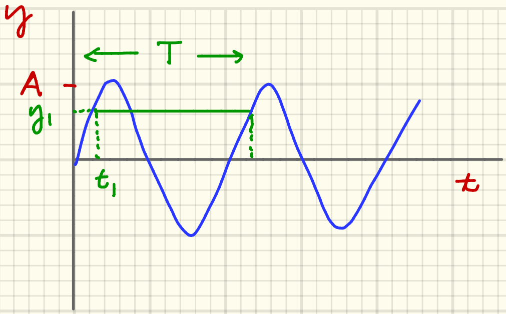
but $y_1 = y_2$ so:

$$x_1 - v t_1 = x_2 - v t_2$$

$$v = \frac{x_2 - x_1}{t_2 - t_1} > 0 \quad \text{since } x_2 > x_1$$

the − sign ⇒ wave moving RIGHT

" + " LEFT



$@\ x = x_1$

at a given x_1, the wave is moving $\updownarrow$

T = period, time for 1 vibration

$\frac{1}{T}$ = rate of vibration = frequency.

$f = \frac{1}{T}$ so $\lambda = vT$

$v = f\lambda \equiv v_p$ "phase velocity"

and

$$y(x,t) = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{v}{\lambda} t \right)$$

$$= A \sin 2\pi \left(\frac{x}{\lambda} - ft \right)$$

standard definitions:

$k = \frac{2\pi}{\lambda}$ "wave number" L^{-1}

$\omega = 2\pi f$ "angular frequency" T^{-1}

$$y(x,t) = A \sin (kx - \omega t)$$

$v_p = f\lambda = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k}$ phase velocity

advance $\delta = \frac{\pi}{2}$ $\quad y(x,t) = A \sin (kx - \omega t + \pi/2) = A \cos (kx - \omega t)$

2 waves : Superposition

$$y = y_1 + y_2$$

$$= A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$ same A, different λ & T ---

$$\cos a + \cos b = 2\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}(a+b)$$

$$y = 2A\cos\tfrac{1}{2}\left\{(k_2-k_1)x - (\omega_2-\omega_1)t\right\}\cos\left\{\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right\}$$

$$\Delta k = k_2 - k_1$$

$$\Delta\omega = \omega_2 - \omega_1$$

$$\bar{k} = \frac{k_1+k_2}{2}$$

$$\bar{\omega} = \frac{\omega_1+\omega_2}{2}$$

$$y = 2A\underbrace{\cos\left\{\frac{\Delta k}{2}x - \frac{\Delta\omega t}{2}\right\}}_{\text{inside this envelope}}\underbrace{\cos\left\{\bar{k}x - \bar{\omega}t\right\}}_{\text{traveling wave}}$$

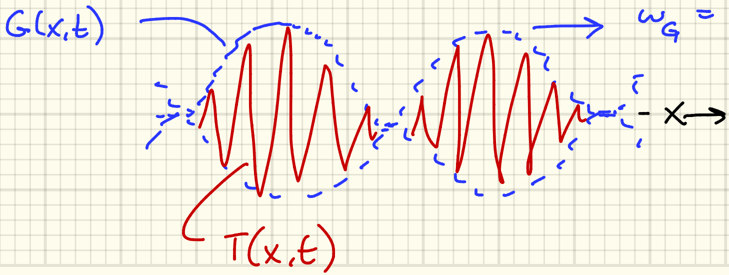
inside this envelope

traveling wave

$$y = 2A\cos\Big\{\underbrace{\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t}_{\text{inside this envelope}}\Big\}\cos\Big\{\underbrace{\bar{k}x - \bar{\omega}t}_{\text{traveling wave}}\Big\} = G(x,t)\,T(x,t)$$

The envelope has a $v_G = \dfrac{\Delta\omega}{\Delta k}$ ... the original waves, y_1 and y_2 continue to have their original $v_{p_1} = \dfrac{\omega_1}{k_1}$ and $v_{p_2} = \dfrac{\omega_2}{k_2}$

The whole system moves with modulation

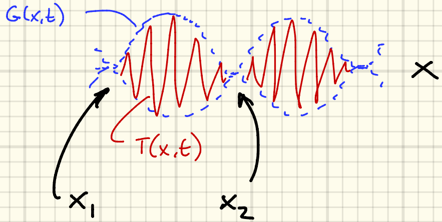


$\omega_G = \dfrac{\Delta\omega}{\Delta k}$

G is called a "wave packet"

Mathematica

$$y = \underbrace{2A\cos\left\{\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right\}}_{G}\ \underbrace{\cos\left\{\bar{k}x - \bar{\omega}t\right\}}_{T}$$



For a given <u>time</u>,

the envelope is localized between x_1 and x_2

when

$$\frac{\Delta k}{2}x_2 - \frac{\Delta k}{2}x_1 = \pi \qquad \Delta x = x_2 - x_1$$

$$\Delta k\,\Delta x = 2\pi$$

At a given <u>position</u>

$$\Delta\omega\,\Delta t = 2\pi$$

Fine-grained localization $\Rightarrow$ Δx small $\Rightarrow$ Δk must be large.

Until this point... only 2 waves.

a true wavepacket with clear "edges" and localization requires many.

From $v_G = \frac{\Delta\omega}{\Delta k} \longrightarrow \left.\frac{d\omega}{dk}\right|_{k_0}$

central wavenumber of lots of k's

In general $\omega = k v_p$

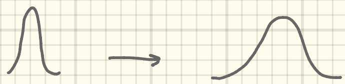
$$v_G = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.v_p\right|_{k_0} + k\left.\frac{dv_p}{dk}\right|_{k_0}$$

velocity depends on wavenumber

... on wavelength

→ dispersion

eg glass $n(\lambda)$



connection between v_G and v_p

A couple of ways ...

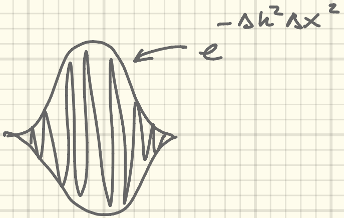
$$y(x,t) = \sum_i A_i \cos(k_i x - \omega_i t)$$ Fourier Series

or for a continuous spectrum

$$y(x,t) = \int \tilde{A}(k) \cos(kx - \omega t)\, dk$$ Fourier Integral

or for practical reasons...

$$y(x,0) = A e^{-\Delta k^2 x^2} \cos(k_0 x)$$ Gaussian wave packet



What about quantum physics you're asking?

$$E = hf = \frac{h\omega}{2\pi} = \hbar\omega \Rightarrow E = \hbar\omega$$

deBroglie assumption $\longrightarrow$ $$p = \frac{h}{\lambda} = \frac{hk}{2\pi} \Rightarrow p = \hbar k$$

Group velocity:

$$v_p = f\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p}$$

$$E = \sqrt{p^2c^2 + m^2c^4}$$

$$v_p = \sqrt{\frac{p^2c^2 + m^2c^4}{p^2}} = c\sqrt{1 + \left(\frac{mc}{p}\right)^2}$$

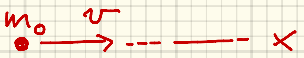
$$v_p = c\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2} = v_p(k)$$

$$v_p = c\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}$$

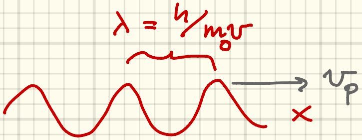
now: $$v_G = v_p\Big|_{k_0} + k\frac{dv_p}{dk}\Big|_{k_0}$$

$$\boxed{v_G = \frac{c}{\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}} = \frac{c^2}{v_p|_{k_0}} \qquad \text{hmm.}}$$

Classical



deBroglie:



$$v_p = f\lambda$$

$$= \frac{E}{\hbar} \cdot \frac{h}{p} \qquad E = hf \qquad p = \frac{h}{\lambda}$$

$$E^2 = p^2c^2 + m_0^2c^4 = (mc^2)^2$$

$$p = m_0 \gamma v = mv$$

$$v_p = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v}$$

BUT WE JUST FOUND

$$\boxed{v_G = \frac{c}{\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2}} = \frac{c^2}{v_p\big|_{k_0}}}$$

hmm.

GROUP VELOCITY = PARTICLE VELOCITY

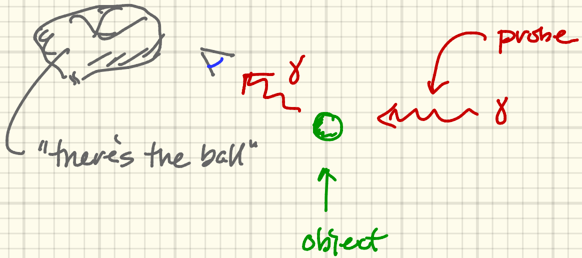
$\Downarrow$

Think of a "particle" with mechanical velocity as represented by a wave packet with group velocity

HOW DO WE KNOW THAT SOMETHING IS... THERE

must "look" at it.

generalized "eyes" -- detectn

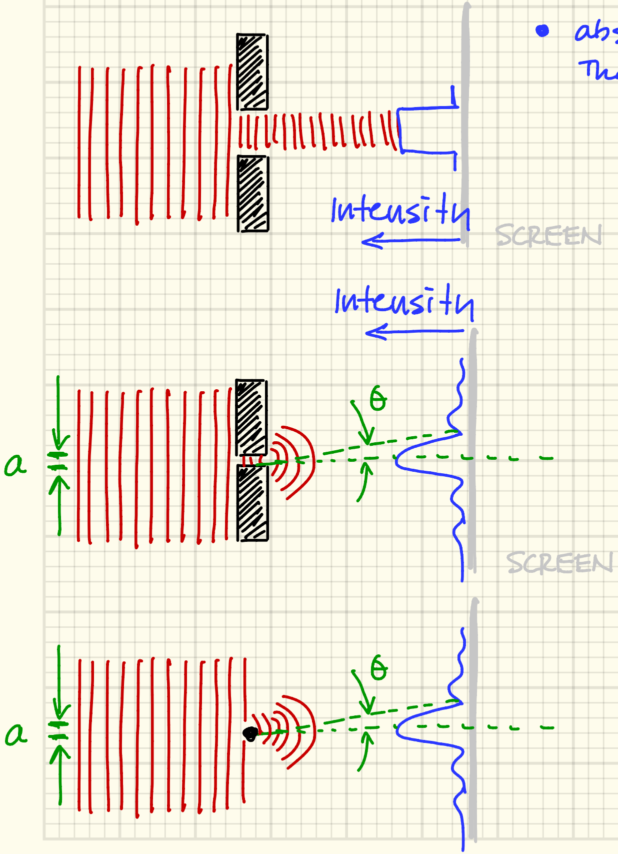


NOW... our object and probe are both particles & waves

imagine object is tiny -- an electron
&
probe is light

HEISENBERG TERRITORY

- formally derived with Q.M.
- abstractly imagined with Thought Experiments



$$a \sin \theta_s = m\lambda$$

$$\theta_s = \frac{m\lambda}{a} \longleftarrow \text{small } \lambda,$$

localized →

BUT, hold the 

Light is also a particle

$p = \frac{h}{\lambda}$ ← smaller the λ ... higher the momentum, p

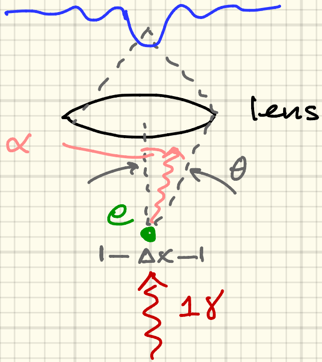
Location of an electron:

How well do you know that it's here

and not here?

using light?

Heisenberg thought experiment



to see it, the scattered γ must be: $-\theta$ to θ -- $\Delta\theta = 2\theta$

$\downarrow$

kicks electron w/ p_x between

$$\Delta p = +\frac{h}{\lambda}\sin\theta \quad \text{to} \quad -\frac{h}{\lambda}\sin\theta$$

physical optics says smallest Δx:

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

$$\Delta x \, \Delta p = \frac{\lambda}{2\sin\theta} \cdot \frac{2h\sin\theta}{\lambda} = h$$

can't use anything less than 1 photon... we're stuck

Heisenberg Uncertainty Principle

$$\Delta p \Delta x \geq \frac{\hbar}{2} \quad \& \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- properties of all waves
- de Broglie makes it seem strange

A few years ago...

pulled over for doing 105 mph*

State police radar: ~20 GHz, $\lambda \sim 14$ cm

Could he resolve my speed? Really?

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \qquad \Delta p = m \Delta v$$

$$m \hat{=} 1500 \text{ kg}$$

$$\Delta p = \frac{\hbar}{2 \Delta x}$$

$$\Delta v m = \frac{\hbar}{2 m \Delta x} =$$

$$\Delta v = \frac{\hbar}{2 \, (\Delta \lambda)(m)} = \frac{1.05 \times 10^{-34} \text{ J}\cdot\text{s}}{2\,(0.14\text{m})(1500\text{ kg})}$$

$$\Delta v \hat{=} 3.5 \times 10^{-37} \text{ m/s}$$

* it was another black BMW...not me!

Electron in 1^{st} Bohr orbit...

remember, I calculated $v \simeq 2 \times 10^6$ m/s

calculate ΔE

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta p \sim mv$$

$$\Delta x = \frac{\hbar}{2\Delta p} = \frac{\hbar}{2mv} = \frac{1.05 \times 10^{-34}\ J \cdot s}{(2)(9 \times 10^{-31}\ kg)(2 \times 10^6\ m/s)}$$

$$\Delta x = 3 \times 10^{-11}\ m \simeq 0.3\ \mathring{A} \sim a_0$$