5. Quantum Mechanics 1, 2

lecture 20, October 13, 2017

housekeeping

exam 2: Friday, October 27

Exam 1:

- the average was 19/30 (not counting extra credit)
- Watch the blog at 3pm on Sunday for instructions for an opportunity to recoup some points. You'll have 24 hours to accomplish this.

Next Tuesday

- The department has an "Investiture"…I have to be an adult and give a speech down the hall
- I'll figure out some way for the "HW Workshop" to happen.

today

waves…generic

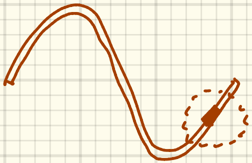
waves…matter

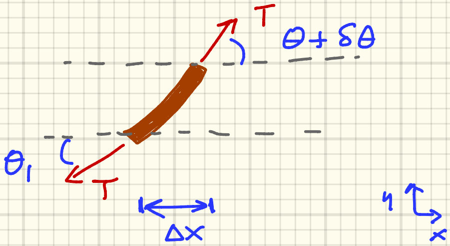
Uncertainty

WAVES

Review (?) of mechanical waves

STRING THEORY ?





Short section: T constant at both ends, but angles slightly different

Vertically: $(F_{net})_y = T\sin(\theta+\delta\theta) - T\sin\theta$

for small θ $\sin\theta \sim \tan\theta = \frac{\partial y}{\partial x}$ → slope at point x y goes up and down in time $\Rightarrow$ transverse velocity $\frac{\partial y}{\partial t}$

$$(F_{net})_y \overset{\sim}{=} T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

which makes you think "derivative" ...

$$\frac{(F_{net})_y}{\Delta x} = T\left\{\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x}\right\}$$

take limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2}$$

The y position is $y(x,t)$...

$$(F_{net})_y = T\,\frac{\partial^2 y}{\partial x^2}\,\Delta x$$

$$(F_{net})_y = T \frac{\partial^2 y}{\partial x^2} \Delta x$$

the mass of that chunk is $m = \underset{\underset{\text{mass/length}}{\uparrow}}{\mu} \Delta x$

From Newton's 2nd:

$$(F_{net})_y = \sum F_y = m a_y$$

$$T \frac{\partial^2 y}{\partial x^2} \Delta x = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

so

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \underset{\uparrow}{\left(\frac{\mu}{T}\right)} \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\left[\frac{\mu}{T_{en}}\right] = \frac{\left[M/L\right]}{\left[\frac{ML}{T_{ime}^2}\right]} = \frac{T_{ime}^2}{L^2} = \left[\frac{1}{v}\right]^2$$

so that term is the inverse of the ↕ velocity

... fr an actual, material wave, its $\left(\frac{\text{mass/length}}{\text{tension}}\right)$

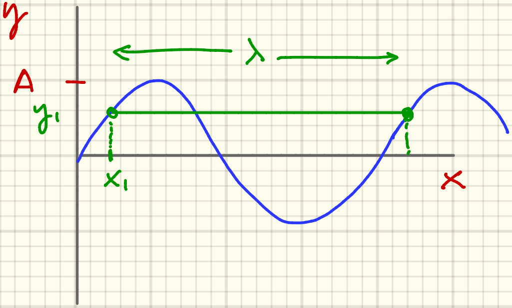
Ta Da:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ is THE WAVE EQUATION

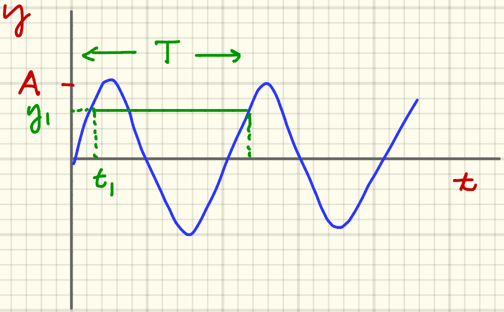
Real solutions: $y(x,t) = A \sin \frac{2\pi}{\lambda}\left(x \pm vt + \delta\right)$

phase depends on where $t = 0$, set $\delta = 0$.

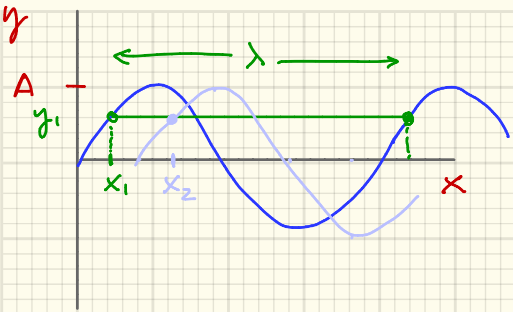
traveling waves $y(x,t) = A \sin \frac{2\pi}{\lambda}\left(x \pm vt\right)$



@ $t = t_1$



@ $x = x_1$



@ $t = t_1$

at some later time, the height position has moved -- choose – sign

$$y_1(x_1, t_1) = A \sin \frac{2\pi}{\lambda}(x_1 - vt_1)$$

to

$$y_2(x_2 t_2) = A \sin \frac{2\pi}{\lambda}(x_2 - vt_2)$$

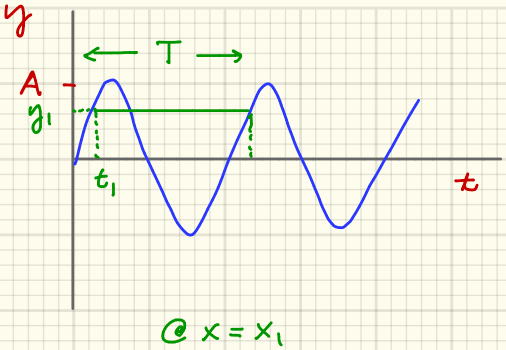
but $y_1 = y_2$ so:

$$x_1 - vt_1 = x_2 - vt_2$$

$$v = \frac{x_2 - x_1}{t_2 - t_1} > 0 \quad \text{since } x_2 > x_1$$

the - sign => wave moving RIGHT

" + " LEFT



at a given x_1, the wave is moving ↕

T = period, time fr 1 vibration

$\frac{1}{T}$ = rate of vibration = frequency.

$f = \frac{1}{T}$ so $\lambda = vT$

$v = f\lambda \equiv v_p$ "phase velocity"

and

$$y(x,t) = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{v}{\lambda} t \right)$$

$$= A \sin 2\pi \left(\frac{x}{\lambda} - ft \right)$$

standard definitions:

$k = \frac{2\pi}{\lambda}$ "wave number" L^{-1}

$\omega = 2\pi f$ "angular frequency" T^{-1}

$v_p = f\lambda = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k}$ phase velocity

$$y(x,t) = A \sin(kx - \omega t)$$

advance $\delta = \frac{\pi}{2}$ $y(x,t) = A \sin(kx - \omega t + \pi/2) = A \cos(kx - \omega t)$

2 waves : Superposition

$$y = y_1 + y_2$$

$$= A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$ same A, different λ & T ---

$$\cos a + \cos b = 2\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}(a+b)$$

$$y = 2A\cos\tfrac{1}{2}\left\{(k_2-k_1)x - (\omega_2-\omega_1)t\right\}\cos\left\{\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right\}$$

$$\Delta k = k_2 - k_1$$

$$\Delta\omega = \omega_2 - \omega_1$$

$$\bar{k} = \frac{k_1+k_2}{2}$$

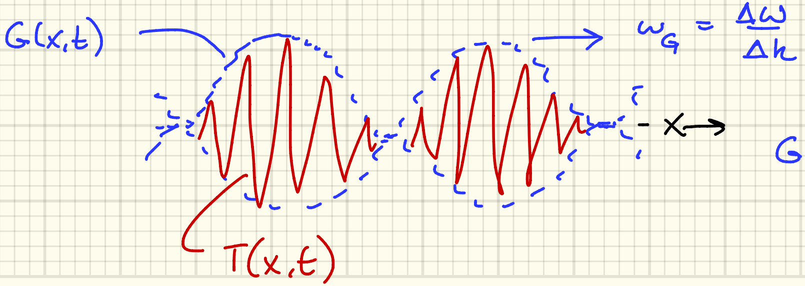
$$\bar{\omega} = \frac{\omega_1+\omega_2}{2}$$

$$y = \underbrace{2A\cos\left\{\frac{\Delta k}{2}x - \frac{\Delta\omega t}{2}\right\}}_{\text{inside this envelope}}\ \underbrace{\cos\left\{\bar{k}x - \bar{\omega}t\right\}}_{\text{traveling wave}}$$

$$y = \underbrace{2A\cos\left\{\frac{\Delta k}{2}x - \frac{\Delta\omega t}{2}\right\}}_{\text{inside this envelope}}\underbrace{\cos\left\{\bar{k}x - \bar{\omega}t\right\}}_{\text{traveling wave}} = G(x,t)\,T(x,t)$$

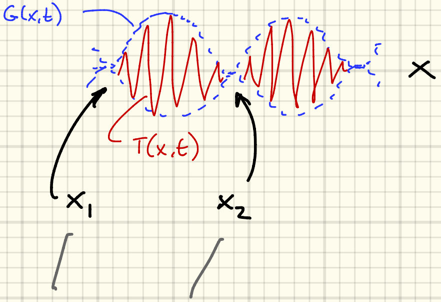
The envelope has a $v_G = \frac{\Delta\omega}{\Delta k}$ ... the original waves, y_1 and y_2 continue to have their original $v_{p_1} = \frac{\omega_1}{k_1}$ and $v_{p_2} = \frac{\omega_2}{k_2}$

The whole system moves with modulation



G is called a "wave packet"

$$y = \underbrace{2A\cos\left\{\frac{\Delta k}{2}x - \frac{\Delta\omega t}{2}\right\}}_{G}\underbrace{\cos\left\{\bar{k}x - \bar{\omega}t\right\}}_{T}$$



For a given time,

the envelope is localized between x_1 and x_2

when $$\frac{\Delta k}{2}x_2 - \frac{\Delta k}{2}x_1 = \pi \qquad \Delta x = x_2 - x_1$$

$$\Delta k\,\Delta x = 2\pi$$

At a given position

$$\Delta\omega\,\Delta t = 2\pi$$

Fine-grained localization ⇒ Δx small ⇒ Δk must be large.

Until this point... only 2 waves.

a true wavepacket with clear "edges" and localization requires many.

From $v_G = \frac{\Delta \omega}{\Delta k} \longrightarrow \left.\frac{d\omega}{dk}\right|_{k_0}$

↑ central wavenumber of lots of k's

In general $\omega = k v_p$

$$v_G = \left.\frac{d\omega}{dk}\right|_{k_0} = v_p\Big|_{k_0} + k \left.\frac{dv_p}{dk}\right|_{k_0}$$

connection between v_G and v_p

↑ velocity depends on wavenumber

... on wavelength

→ dispersion

eg glass $n(\lambda)$

A couple of ways ...

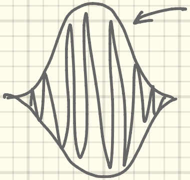
$$y(x,t) = \sum_i A_i \cos(k_i x - \omega_i t)$$ Fourier Series

or for a continuous spectrum

$$y(x,t) = \int \tilde{A}(k) \cos(kx - \omega t)\, dk$$ Fourier Integral

or for practical reasons...

$$y(x,0) = A e^{-\Delta k^2 x^2} \cos(k_0 x)$$ Gaussian wave packet


$e^{-\Delta k^2 \Delta x^2}$

What about quantum physics you're asking?

$$E = hf = \frac{h\omega}{2\pi} = \hbar\omega \quad \Rightarrow \quad E = \hbar\omega$$

deBrogle assumption $\longrightarrow$

$$p = \frac{h}{\lambda} = \frac{h\,k}{2\pi} \quad \Rightarrow \quad p = \hbar k$$

Group velocity:

$$v_p = f\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p}$$

$$E = \sqrt{p^2c^2 + m^2c^4}$$

$$v_p = \sqrt{\frac{p^2c^2 + m^2c^4}{p^2}} = c\sqrt{1 + \left(\frac{mc}{p}\right)^2}$$

$$v_p = c\sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2} = v_p(k)$$

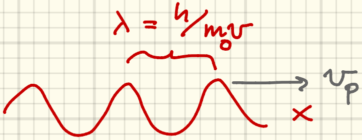
$$v_p = c\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}$$

now: $$v_G = v_p\Big|_{k_0} + k\frac{dv_p}{dk}\Big|_{k_0}$$

$$\boxed{v_G = \frac{c}{\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}} = \frac{c^2}{v_p\big|_{k_0}} \qquad \text{hmm.}}$$

Classical

deBroglie:



$$v_p = f\lambda$$

$$= \frac{E}{\hbar} \cdot \frac{h}{p} \qquad E = hf \qquad p = \frac{h}{\lambda}$$

$$E^2 = p^2c^2 + m_0^2c^4 = (mc^2)^2$$

$$p = m_0\gamma v = mv$$

$$v_p = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v}$$

BUT WE JUST FOUND

GROUP VELOCITY = PARTICLE VELOCITY

⇓

$$\boxed{v_G = \frac{c}{\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}} = \frac{c^2}{v_p\big|_{k_0}}}$$

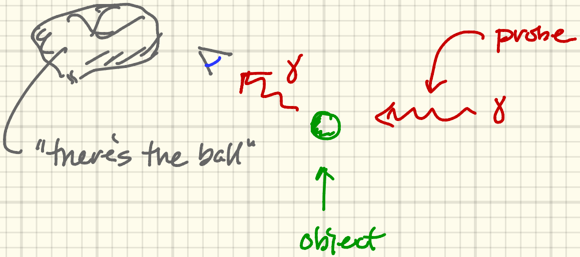
hmm.

Think of a "particle" with mechanical velocity as represented by a wave packet with group velocity

HOW DO WE KNOW THAT SOMETHING IS... THERE

must "look" at it.

↑ generalized "eyes" -- detectn



NOW... our object and probe are both particles & waves

imagine object is tiny -- an electron
&
probe is light

I'm now uncertain.

Heisenberg, in the best Einsteinian tradition, asked a simple question:

what's involved in measuring something...?

It’s all about the deBroglie relation

relating the wavelength of a quantum object

to its momentum

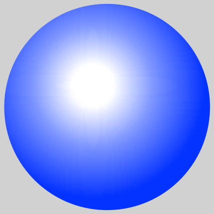
$$\lambda = \frac{h}{p}$$

it was hard enough

for photons

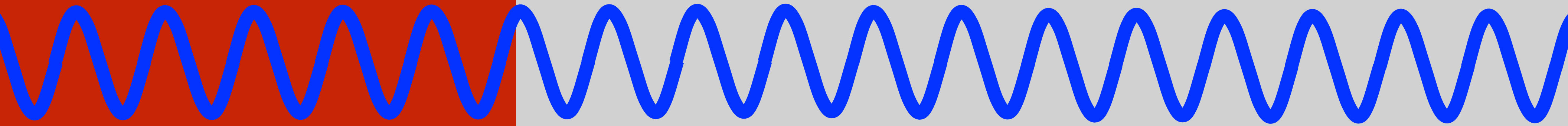
but for an electron?

A particle is HERE:



$$p = mv$$

A wave is EVERYWHERE:



The deBroglie hypothesis:

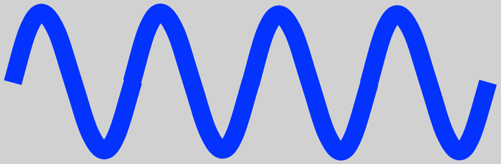
of given momentum

<u>also</u> has

a single wavelength

$$p = \frac{h}{\lambda}$$

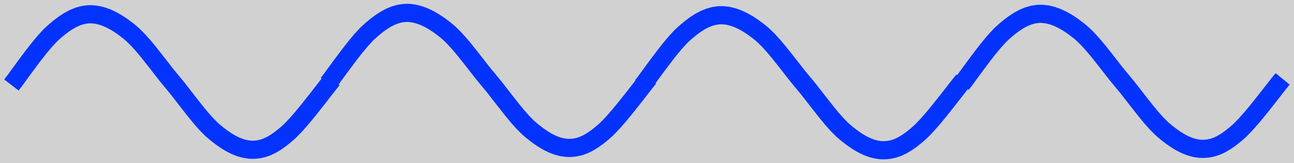
immediate implications



$$p_1 = \frac{h}{\lambda_1}$$

wavelength and momentum are inversely linked

immediate implications

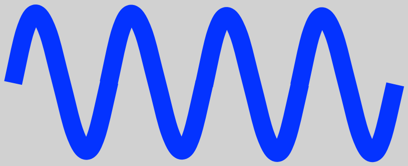


$$p_2 = \frac{h}{\lambda_2}$$

$$p_2 < p_1$$

long wavelength: low momentum

immediate implications



$$p_3 = \frac{h}{\lambda_3}$$

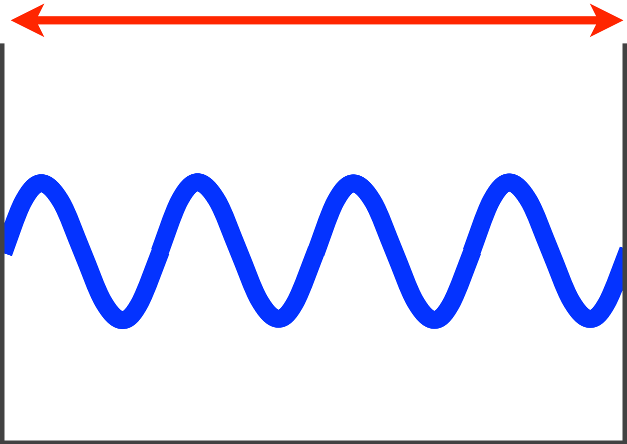
$$p_3 > p_1$$

short wavelength: high momentum

suppose we trap

an electron

Where's the electron?



somewhere here:

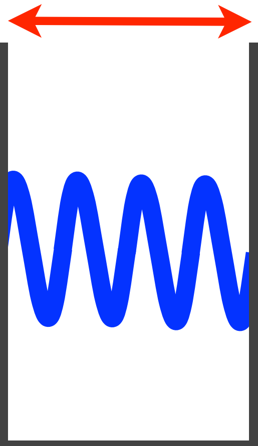
how to locate it better?

suppose we trap

an electron

Where's the electron?

somewhere here:



make the trap smaller

$$p = \frac{h}{\lambda}$$

The wavelength is shorter...
So the momentum is higher!

an inevitable trade-off

in order to make the location more precise

you pay the price that its **speed is higher**

Heisenberg Uncertainty Principle

the Heisenberg Uncertainty Principle

was from 26 year old Werner Heisenberg

an enigma

inventor of many important concepts

did he save the west from a German nuclear bomb?

or the opposite?

Werner Heisenberg 1901-1976

measuring something...

you have to "look" at it

by eye or some external, intermediate probe

remember for waves what determines the scale?

wavelength

What if the object is atomic sized or smaller? ... what is it to "look"??

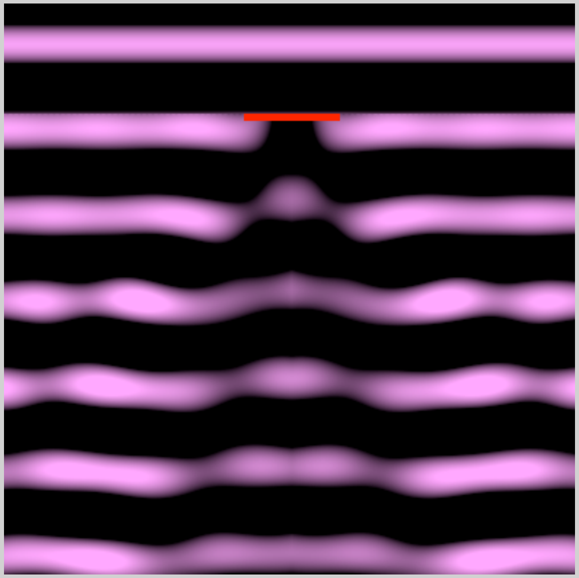
Heisenberg Uncertainty ... really!

how do you measure the trajectory of an object?

look at it in Time

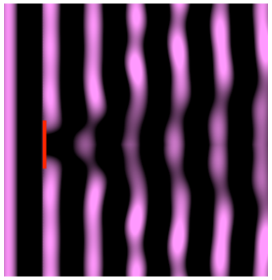
→ bounce light off it

Sweet spot for identifying an object:
need λ ~ size of the object



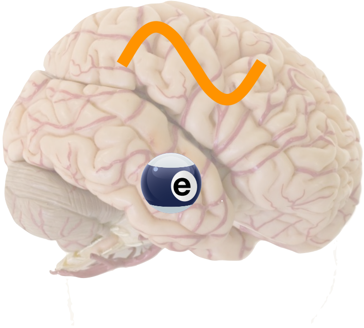
uncertainty - sometimes called “indeterminancy”

Try to “see” and electron.
Electrons are small.
So...need light wavelength small.



$$\}\,\Delta x \sim \lambda$$

So, make λ small to reduce Δx



Gedankenexperiment

Photon diffracts by the electron “barrier” and blurs the electron position by about the amount of the photon wavelength

But, $p = \frac{h}{\lambda}$ makes p large!

$$\Delta p \sim \frac{h}{\Delta x}$$

so now knowledge of the momentum is blurred

$$\Delta p \Delta x \sim h$$

`there is`

NO WAY to beat it in any of these measurement scenarios

the inverse relation between p and λ messes with you every time

$$p = \frac{h}{\lambda}$$

but here’s the hard part

what does it mean?

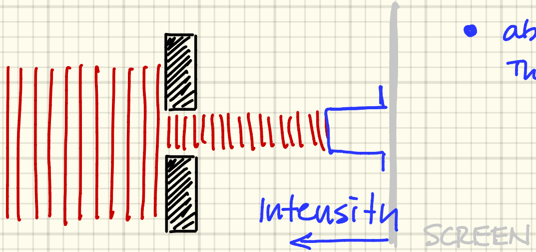
the inability to determine position or momentum to arbitrary precision

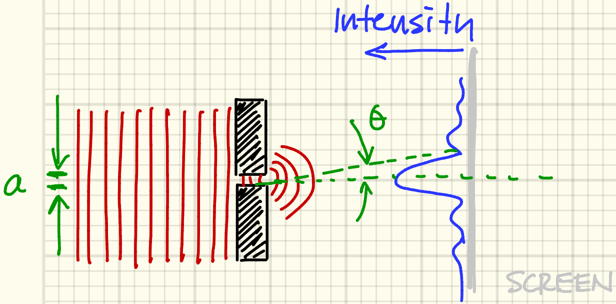
is not about poor instruments

It. Is. About. Nature.

HEISENBERG TERRITORY

- formally derived with Q.M.
- abstractly imagined with Thought Experiments

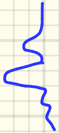


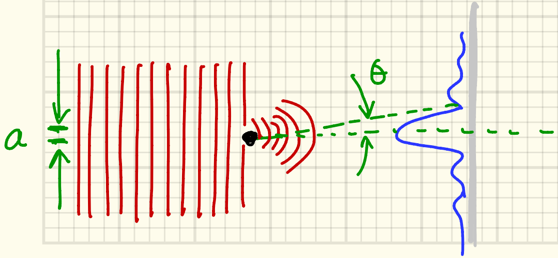


$$a \sin\theta_s = m\lambda$$

$$\theta_s = \frac{m\lambda}{a} \longleftarrow \text{small } \lambda,$$

localized →





BUT, hold the

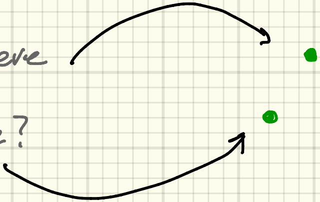
Light is also a particle

$$p = \frac{h}{\lambda}$$ ← smaller the λ ... <u>higher</u> the momentum, p

Location of an electron:

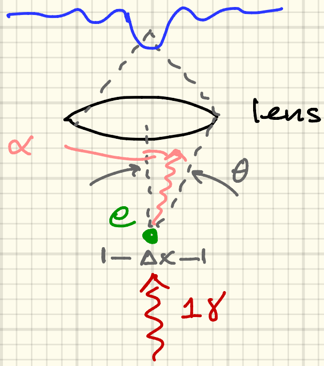
How well do you know that it's here

and not here?



using light?

Heisenberg thought experiment



to see it, the scattered γ must be: $-\theta$ to θ -- $\Delta\theta = 2\theta$

↓

kicks electron w/ p_x between

$$\Delta p = +\frac{h}{\lambda}\sin\theta \quad \text{to} \quad -\frac{h}{\lambda}\sin\theta$$

physical optics says smallest Δx:

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

$$\Delta x\,\Delta p = \frac{\lambda}{2\sin\theta}\cdot\frac{2h\sin\theta}{\lambda} = h$$

can't use anything less than 1 photon... we're stuck

Heisenberg Uncertainty Principle

$$\Delta p \Delta x \geq \frac{\hbar}{2} \quad \& \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- properties of all waves
- de Broglie makes it seem strange